Rolling the dice – exploring different approaches to probability with primary school students

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This paper focuses on a probability project and shows how primary school students develop probabilistic thinking according to different approaches to probability. In the project students explore the rolling of "odd dice" and compare the winning chances of these dice. By experimenting with the odd dice and comparing their structure to achieve insight into winning strategies the students have to deal with different approaches to probability, especially the subjective and frequency interpretations. A specific work-flow for processing such projects is presented.

Keywords: Probability, frequentist approach, subjective approach, learning environment, classroom project.

INTRODUCTION

This study wants to make a contribution on how teaching of probability in primary school could be laid out, by introducing a mathematical project on "odd dice", which is open for an experimental approach and rich discoveries on probabilities beyond the standard material in stochastics. Furthermore a specific project work-flow for the design of the classroom interaction and for the analysis of the learning process as a diagnostic tool is presented.

THEORETICAL FRAMEWORK: APPROACHES TO PROBABILITY

There are mainly three different approaches to the concept of probability which are crucial for teaching probability at primary schools and aiming at a broad and reasonable understanding of probability for students. Since you only have limited access to an axiomatic interpretation of probability in school mathematics it is important to show alternative interpretations which connect to students' previous knowledge and everyday experiences, even appreciating subjective ideas and conceptions and relate them to mathematical views on probability.

Subjective approach

The subjective approach to probability is the one you will encounter first when teaching probability in school mathematics. "We identify probabilities with degrees of confidence, or credences, or 'partial' beliefs of suitable agents" (Hájek, 2012). Almost all children gathered experiences with statements on probability in the context of chances in games e.g. rolling six dots with dice in the game "Ludo". The subjective approach is fully loaded with individual experiences, naïve ideas and with personal preferences to the point of superstitious beliefs like lucky numbers (see Büchter et al., 2005). Children hold certain ideas and conceptions and that is what teachers have to deal with for successfully teaching stochastics. By setting up activating learning environments such pre-experiences of students could be made explicit for the learning process. The teaching problem is that some of the students' conceptions do not match with the mathematical concepts. To overcome this gap it is necessary to get students into a reflection on concepts.

Frequentist approach

The frequentist approach to probability will help to develop a broader understanding of probabilistic processes. It defines a probability of an event as the limit of its relative frequency in a large number of trials, according to the law of large numbers. In primary schools one often deals just with counting absolute frequency in relation to a fixed number of trials, e.g., rolling the dice 100 times and then counting the number of occurrences of the pips. This empirical approach suits well with activating learning environments in which students are able to determine probabilities by random experiments. Students will evaluate their experiments with tally tables.

Classic approach

At last the classic approach to probability will be considered as a mathematical sophisticated approach which has a profound theoretical meaning but is only accessible in a limited way in primary schools. The probability of an event is given by its ratio of the number of cases favorable to it, to the number of all cases possible. This requires that none of the cases occur more than any other i.e. all cases are equally possible. This approach goes back to a definition by Laplace from 1814 in his Philosophical Essay on Probabilities. Since fractions and rational numbers are only available in a very limited way in primary school it will not be possible to apply this approach in-depth. But the basic idea could be used to count the favorable cases and compare them to a fixed number of possible cases. A special case of the classic approach is the geometric interpretation of probability, e.g. looking at the faces of the dice and – assuming that all faces are the same size – the probability for each face will be the same.

"ODD DICE" – A PROBABILITY PROJECT

The teaching material "Spürnasen Mathematik" [mathematics sleuths] consists of a box with mathematical projects on arithmetic, geometry and stochastics aiming at an open, activity-oriented learning process, accompanied with working books for a systematical training and learning process. To ensure rich experiences with mathematics and linking-up with situations of everyday life the projects are starting always with a hands-on activity for children, followed up by a step by step systematization and formalization with mathematical tools and mathematical language. This specific approach is chosen to increase the motivation to deal with mathematical contents and tools, and to see the potential of mathematics to describe everyday life situations, to solve real life problems and to provide a language to communicate and compare information and data. The mathematical tasks and the project work are open according to Peschel (2007) in the classroom format and social form in which the children will work on the projects (e.g. most project tasks could be carried out individually, pair or group work), the organization of the learning process (e.g. the children can choose from different materials to work with), and the conceptual opening (e.g. the children are encouraged to solve the problems on their competence level and in their way of mathematical thinking). Therefore the projects have a high potential for differentiation in the learning process and the resulting products. This classroom research report will focus on the project named "Only by Chance" classified as a stochastics project. This projects aims for an exploration of dealing with chances and probabilities in different contexts.

One part of the project looks at "odd dice" and the odds to win by playing dice and the corresponding probabilities. The odd dice are given by their cube nets (see Figure 1). These four coloured dice are an adaption of the "Miwin dice" (Winkelmann, 2012) and do not show the usual dots of a dice and equally probable occurrence of each side of the dice, but a special arrangement of the numbers 0 to 6 as shown in Figure 1.

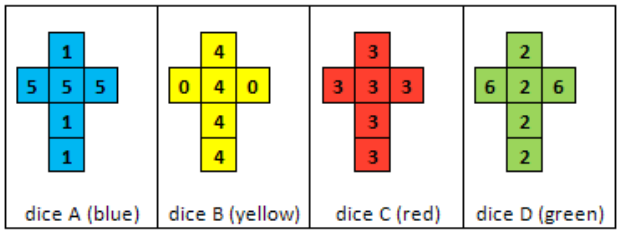

Figure 1: Cube nets of the odd dice for the probability project

In the following I will outline the project tasks for the students, some of the learning outcome objectives and possible solutions of the problems. (Remark: In the teaching material "Spürnasen Mathematik" all the tasks are gathered on a task instruction card. Since the material is in German the original card is not shown.)

Task 1

The first task of the project asks the students to explore with prepared odd dice, which numbers on the dice will occur most frequently. After making a guess, the students roll the dice and take track of the occurrences with a tally sheet (see Figure 2). This will help students to get started with the project and become more familiar with the odd dice, since they probably haven't been working with such dice beforehand. The students learn to estimate the possible outcomes of throwing the dice, and are encouraged to experiment and set up a series of dice rolling. By documenting their results in tally sheets they will learn a specific technique of data representation.

Figure 2: Suggestion for documenting the occurrences

Task 2

The second task is to roll the dice in a little game of dice for two students using the red and green dice. The student throwing the higher number will win and gets one point. The game will stop when one student has gained 10 points. Again the students should make an assumption first and then use a tally list to write down the gaming results, and observe which dice is winning the game. It is necessary to give the students enough time to explore the different dice and to analyze which dice is better. The game and a systematic representation of the game result will lead to a reasonable good assumption of the better dice.

Task 3

In the following task the students should play the dice game over and over again with two different dice in order to find "the best" dice, using one of the various suggested documentation styles for their assumptions and their results. With a systematic approach for testing all combinations of dice – this is encouraged by the proposed documentation forms – the students have to explore, communicate and argue to find the best dice of all. A demonstrative and helpful tool for comparing the dice are the tables showing for each pair of dice the possible events and marking the winning dice (see Figure 3).

These tables give insight into the winning probability of each pair of odd dice. For example have a look at the top left corner of Figure 3. There you can see dice C (red) and dice D (green) with all possible cases of rolling dice results. Each table cell states (by colour and letter) which dice is the winning dice for a certain case. You can clearly see, that the red dice will win in more cases than the green dice (to be precise in 24 out of 36 cases), thus making the red dice the better choice for the game. Evaluating all tables you will end up with a diagram shown in Figure 4. There is no dice with exclusively outgoing or incoming arrows in the diagram, therefore none of the dice could be considered 'the best', but you always find one dice which is – on the long run – better than (or at least as good as) a certain chosen dice. This will not imply you are winning in every turn of the game, but if you as second player in the game choose the "right" dice according to the diagram you will enlarge your winning odds and on the long run with many follow-ups of the game win the game more often.

Task 4

By using proposed forms of documentation the students will result in a good overview of the winning odds of the dice and be able to answer the last task. In the final task the students are requested to discuss a given statement of student Jonas (illustrated in the teaching material), that you will always win the game if you take the second turn to pick one dice, and to argue on the question, why these dice are called "odd". Here the students are trained to take a close look at

dice D (green) \ dice C (red)	3	3	3	3	3	3
2	C	C	C	C	C	C
2	C	C	C	C	C	C
2	C	C	C	C	C	C
2	C	C	C	C	C	C
6	D	D	D	D	D	D
6	D	D	D	D	D	D

dice A (blue) \ dice B (yellow)	0	0	4	4	4	4
1	A	A	B	B	B	B
1	A	A	B	B	B	B
1	A	A	B	B	B	B
6	A	A	A	A	A	A
6	A	A	A	A	A	A
6	A	A	A	A	A	A

dice A (blue) \ dice C (red)	3	3	3	3	3	3
1	C	C	C	C	C	C
1	C	C	C	C	C	C
1	C	C	C	C	C	C
5	A	A	A	A	A	A
5	A	A	A	A	A	A
5	A	A	A	A	A	A

dice D (green) \ dice A (blue)	1	1	1	5	5	5
2	D	D	D	A	A	A
2	D	D	D	A	A	A
2	D	D	D	A	A	A
2	D	D	D	A	A	A
6	D	D	D	D	D	D
6	D	D	D	D	D	D

dice D (green) \ dice B (yellow)	0	0	4	4	4	4
2	D	D	B	B	B	B
2	D	D	B	B	B	B
2	D	D	B	B	B	B
2	D	D	B	B	B	B
6	D	D	D	D	D	D
6	D	D	D	D	D	D

dice B (yellow) \ dice C (red)	3	3	3	3	3	3
0	C	C	C	C	C	C
0	C	C	C	C	C	C
4	B	B	B	B	B	B
4	B	B	B	B	B	B
4	B	B	B	B	B	B
4	B	B	B	B	B	B

Figure 3: Comparison of the odd dice

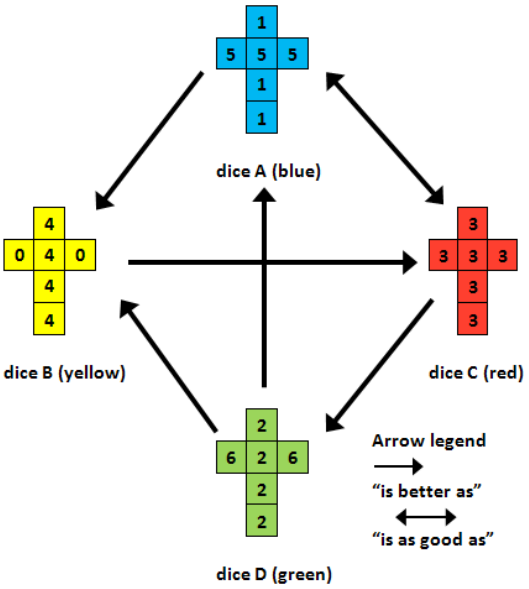

Figure 4: Diagram showing the relation between the odd dice

the situation: Jonas' remark just states that there will be a reasonable choice for a dice in response to a chosen dice in the first draw. But there is no "best" dice at all. This finding is an explanation for the naming of the odd dice and linking these odd dice to the widely known "rock, paper, scissors" game with an analogous relation between the three possible events (there is always one beating another, but no move to win in every case).

RESEARCH QUESTIONS AND DESIGN OF STUDY

The following empirical findings will describe which approaches to probability are used by students working on the project "odd dice" and how their understanding of probability could be developed to a broader approach by working on the tasks concerned with odd dice.

The probability project learning environment was applied in a primary school at Eitorf near Siegen (Germany) in a third grade class with 23 kids (11 girls, 12 boys, age ranging from 8 to 10 years old, 5 with special needs). The students were chosen because they were used to work on open projects since it is the usual learning environment in this school. This was an important criterion to observe if and to what extent it is possible to make students understand a specific mathematical content like probability in such a learning environment setting. The project "odd dice" was processed by two groups with each four students and a third group with two students in about 2 hours. The student's documents and observations of a trainee teacher (Nelia Kasemir who documented the students' results in her final thesis, see Kasemir, 2013), are the basis for an interpretative approach for the in-depth data analysis. The trainee teacher was very familiar with this class and basically played the role of managing the working and learning process according to the work flow presented below (see Figure 5). Only in the first plateau phase some examples for probabilities of events were presented and in the later plateau phases, students were encouraged to share their intermediate results.

For evaluating these project products and learning and teaching processes, a work flow for mathematical projects of the teaching material "Spürnasen Mathematik" (see Helmerich & Lengnink, 2013, and Lengnink, 2012) could be used, following the "Think,

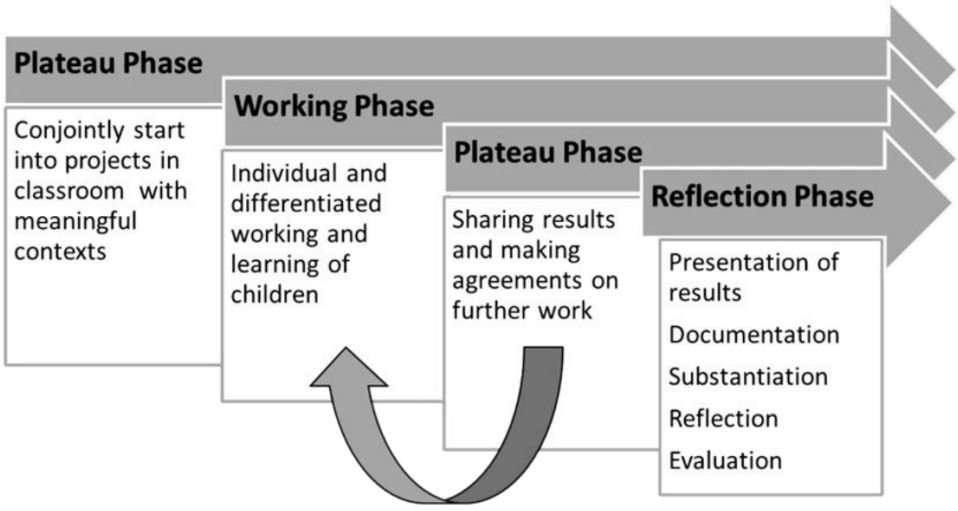

Figure 5: Work flow for mathematical projects in classroom interaction

Pair, Share" method (for example, presented in Barzel et al., 2007, p. 118).

The children are introduced to learning objectives and possible tasks in the project, before starting off with individual and group work. The students of third grade already acquired competence in dealing with probability in the sense of assigning several everyday-life situations to a probability continuum chart reaching from "impossible", over "unlikely" and "rather probable" to "certain". Before the odd dice project they worked on probabilities of usual, regular dice and the urn model and projects on probability of drawing a certain colour out of a package of chocolate beans, and the probability of letters in words and de-coding strategies in Caesar code applications. Within a sharing circle the pre-knowledge was re-activated. The recurrence of plateau phases, where children share their ideas and preliminarily results marks the important issue of reflecting the process and give teachers the opportunity to adjust the children's work for the next working phase.

ANALYSIS OF THE WORKING PROCESS AND RESULTS OF THE STUDENTS

This phase model will be used to analyze the learning process and to show different approaches to probability as seen in the students' documents. In the first phase the students' previous knowledge on and experiences with probabilities are activated by collecting situations in everyday life in which the term 'probability' appear. The students are asked to find a definition for probability and state some events which in their opinion are probable, impossible definite. The most common paraphrase for a "probable event" used, was something "that could happen, but doesn't have to." With this preparation the students start with their group work on different probability projects.

In the second phase the students started off with their work on the project tasks. The students mainly worked together in pairs of two, which were formed at random. With the first task of the "odd dice" project two approaches to probability are activated: the students are supposed to estimate the number of occurrences of the dice faces which activates subjective views. Some students decided to analyze the dice of their favorite colour or the dice with their lucky number on it. The only notion before starting the rolling experiment was on dice C (red) with exclusively threes on each face. It was obvious for the children that this dice will produce the event '3' all the time. Other estimations were not made. With rolling the dice the frequentist approach came into action. It seemed clear that rolling the dice several times will show, which results one can get with each dice. However, some students restricted their tally sheet only to some numbers on the dice, so the actual distribution of the events is not represented but could be extracted from the notes. For example Simon (see Figure 6) rolled all the dice ten times, so you can calculate the number of occurrences of the other dice rolling results. Final remarks on the probabilities or a location in the probability continuum were not made at this stage.

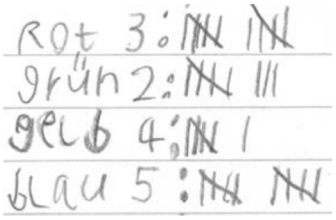

Figure 6: Simon's tally sheet on task 1 (marking the results for the red ("rot"), green ("green"), yellow ("gelb") and blue ("blau") dice)

The students stepped on to task 2 and the comparison of dice. Interestingly some students started the game using both the same coloured dice (see Figure 7). This could be explained by the need to become more familiar with these odd dice and to set up a situation ensuring a level playing field. In these cases the students achieved as expected narrow outcomes in their game.

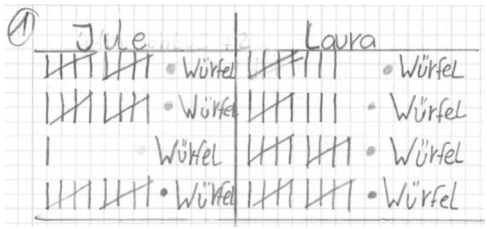

Figure 7: The table of Jule and Laura ("Würfel" means dice)

To overcome this strategy and to get to the actual task it was necessary to compare the first results in a plateau phase. Thereby Jule and Laura were encouraged to proceed with the comparison of dice. In Figure 8 the results of the comparison of the blue and the yellow as well as the red and green dice are shown in a conjoint table.

In this task the difficulty occurred that the results of the game did not represent the expected insight into the relation of the dice since the game stopped

Figure 8: Jule and Laura playing the dice game ("blau" = blue, "orange" = red, "gelb" = yellow, "grün" = green)

with one student gaining ten points. To get over this problem, it might have been helpful to have a systematic look at the dice and their faces and draw tables as shown in Figure 3. Due to the limited time for the project during class this was not worked out in detail. But the students nevertheless tried to figure out whether there is always a dice for winning the game like Jonas suggested. In this process it is interesting to state that the students did not carry out the systematic comparison according to the classic approach to probability but fall back on subjective views. Looking at Simon's documentation of task 4 (see Figure 9) reveals that Simon (in the table abbreviated to "Sim") plays the game with Saheb (short "Sah") by choosing a pair of dice, roll them, record the result and play again with a new pair of dice. This approach to focus on the single outcome of the game could be characterized as a uni-structural thinking (Shaughnessy, 2007; Watson & Kelly, 2004). Thinking in relations and an understanding enriched by relational conceptions was not achieved of Simon. This emphasizes the challenge of teaching probability, to master the mental step from a subjective approach to the more elaborated classic approach of probability.

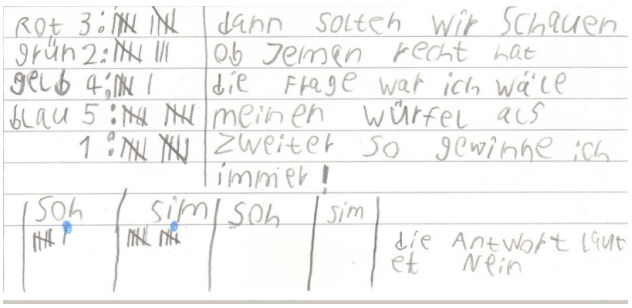

Figure 9: Simon's argumentation on task 4: "then we should have a look if someone is right: the question was: I choose my dice second, thus I will always win! The answer is no."

DISCUSSION AND CONCLUSION

Such mathematical projects like the one on odd dice make it possible to let students explore different approaches to probability in an activating, experimental way. The project even shows the potential not only to stick to the subjective and frequency interpretations of probability but to merge these aspects to the classic approach. The broader view on probability is especially inherent in the plateau phases when students have the opportunity to share their work achieved so far and could be enriched with some new ideas and strategies, also from the teacher, for further project work. In higher grades this project on odd dice could be unfold to a broader mathematical analysis of the odd dice using the classic approach to probability and calculating and comparing the probability for each number on the dice. This might be an approach to an extension of the probability concept to non-Laplacian experiments and the law of large numbers, since in the game situation with a limited number of dice rolls the experiment does not always lead to the theoretically expected outcomes. The project is a contribution to the fundamental mathematical idea of "data and chance" as it is stated in the national standards for mathematics teaching in Germany (e.g., KMK, 2004). It combines the idea of data collection by rolling the dice with the ides of chance by investigating the probabilities of the dice. This study was the starting point for a broader investigation of mathematical ideas in stochastics in primary school teaching. It just gives some insight on the thinking and reasoning of primary school students, but shows the potential of such projects for diagnostic approaches in research of learning processes.

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