Product commuting maps with the λ-Aluthge transform

Fadil Chabbabi

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Université Lille1, UFR de Mathématiques,
Laboratoire CNRS-UMR 8524 P. Painlevé, 59655 Villeneuve d'Ascq Cedex, France

Abstract

Let H and K be two Hilbert spaces and $\mathcal{B}(H)$ be the algebra of all bounded linear operators from H into itself. The main purpose of this paper is to obtain a characterization of bijective maps $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ satisfying the following condition

$$\Delta_\lambda(\Phi(A)\Phi(B)) = \Phi(\Delta_\lambda(AB)) \quad forall \;\; A, B \in \mathcal{B}(H),$$

where $\Delta_\lambda(T)$ stands the λ-Aluthge transform of the operator $T \in \mathcal{B}(H)$.

More precisely, we prove that a bijective map Φ satisfies the above condition, if and only if $\Phi(A) = UAU^*$ for all $A \in \mathcal{B}(H)$, for some unitary operator $U : H \to K$.

Keywords: Normal, Quasi-normal operators, Polar decomposition, λ-Aluthge transform.

1. Introduction

Let H and K be two complex Hilbert spaces and $\mathcal{B}(H, K)$ be the Banach space of all bounded linear operators from H into K. In the case $K = H$, $\mathcal{B}(H, H)$ is simply denoted by $\mathcal{B}(H)$ which is a Banach algebra. For $T \in \mathcal{B}(H, K)$, we set $\mathcal{R}(T)$ and $\mathcal{N}(T)$ for the range and the null-space of T, respectively. We also denote by $T^* \in \mathcal{B}(K, H)$ the adjoint operator of T.

The spectrum of an operator $T \in \mathcal{B}(H)$ is denoted by $\sigma(T)$ and $W(T)$ is the numerical range of T.

An operator $T \in \mathcal{B}(H, K)$ is a *partial isometry* when T^*T is an orthogonal projection (or, equivalently $TT^*T = T$). In particular T is an *isometry* if $T^*T = I$, and *unitary* if T is a surjective isometry.

The polar decomposition of $T \in \mathcal{B}(H)$ is given by $T = V|T|$, where $|T| = \sqrt{T^*T}$ and V is an appropriate partial isometry such that $\mathcal{N}(T) = \mathcal{N}(V)$ and $\mathcal{N}(T^*) = \mathcal{N}(V^*)$.

The Aluthge transform introduced in [1] as $\Delta(T) = |T|^{\frac{1}{2}} V |T|^{\frac{1}{2}}$ to extend some properties of hyponormal operators. Later, in [11], Okubo introduced a more general notion called $\lambda-$Aluthge transform which has also been studied in detail.

For $\lambda \in [0, 1]$, the $\lambda-$Aluthge transform is defined by,

$$\Delta_\lambda(T) = |T|^\lambda V |T|^{1-\lambda}.$$

Notice that $\Delta_0(T) = V|T| = T$, and $\Delta_1(T) = |T|V$ which is known as Duggal's transform. It has since been studied in many different contexts and considered by a number of authors (see for instance, [2, 3, 7, 8, 9, 12] and some of the references there). The interest of the Aluthge transform lies in the fact that it respects many properties of the original operator. For example,

$$\sigma_*(\Delta_\lambda(T)) = \sigma_*(T), \text{ for every } \;\; T \in \mathcal{B}(H), \tag{1}$$

where σ_* runs over a large family of spectra. See [7, Theorems 1.3, 1.5].

Another important property is that $Lat(T)$, the lattice of T-invariant subspaces of H, is nontrivial if and only if $Lat(\Delta(T))$ is nontrivial (see [7, Theorem 1.15]).

Email address: Fadil.Chabbabi@math.univ-lille1.fr (Fadil Chabbabi)

Recently in [4], F.Bothelho, L.Molnár and G.Nagy studied the linear bijective mapping on Von Neumann algebras which commutes with the λ-Aluthge transforms. They focus of bijective linear maps such that

$$\Delta_\lambda(\Phi(T)) = \Phi(\Delta_\lambda(T)) \text{ for every } T \in \mathcal{B}(H).$$

We are concerned in this paper with the more general problem of product commuting maps with the λ-Aluthge transform in the following sense,

$$\Delta_\lambda(\Phi(A)\Phi(B)) = \Phi(\Delta_\lambda(AB)) \text{ for every } A, B \in \mathcal{B}(H), \tag{2}$$

for some fixed $\lambda \in]0, 1[$.

Our main result gives a complete description of the bijective map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ which satisfies Condition (2) and is stated as follows.

Theorem 1.1. *Let H and K be a complex Hilbert spaces, with H of dimension greater than 2. Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be bijective. Then,*

Φ satisfies (2), if and only if, there exists an unitary operator $U : H \to K$ such that

$$\Phi(A) = UAU^* \quad \textit{for all } A \in \mathcal{B}(H).$$

Remark 1.1. (1) *In one dimensional, the result of Theorem 1.1 fails, as given in the following example: let the map $\Phi : \mathbb{C} \to \mathbb{C}$ defined by*

$$\Phi(z) = \begin{cases} \frac{1}{z} & \textit{if } z \neq 0, \\ 0 & \textit{if } z = 0. \end{cases}$$

Clearly Φ is bijective and satisfies (2), but it is not additive.

(2) *The map Φ considered in our theorem is not assumed to satisfy any kind of continuity. However, an automatic continuity is obtained as a consequence.*

The proof of Theorem 1.1 is stated in next section. Several auxiliary results are needed for the proof and are established below.

2. Proof of the main theorem

We first recall some basic notions that are used in the sequel. An operator $T \in \mathcal{B}(H)$ is normal if $T^*T = TT^*$, and is quasi-normal, if it commutes with T^*T (i.e. $TT^*T = T^*T^2$), or equivalently $|T|$ and V commutes. In finite dimensional spaces every quasi-normal operator is normal. It is easy to see that if T is quasi-normal, then T^2 is also quasi-normal, but the converse is false as shown by nonzero nilpotent operators .

Also, quasi-normal operators are exactly the fixed points of Δ_λ (see [7, Proposition 1.10]).

$$T \quad \text{quasi-normal} \quad \Longleftrightarrow \Delta_\lambda(T) = T. \tag{3}$$

An idempotent self adjoint operator $P \in \mathcal{B}(H)$ is said to be an orthogonal projection. Clearly quasi-normal idempotents are orthogonal projections.

Two projections $P, Q \in \mathcal{B}(H)$ are said to be orthogonal if $PQ = QP = 0$ and we denote $P \perp Q$. A partial ordering between orthogonal projections is defined as follows,

$$P \leq Q \text{ if } PQ = QP = P.$$

We start with the following lemma, which gives the "only if" part in our theorem. It has already been mentioned in other papers in the case $H = K$ (see [3], for example). We give the proof for completeness.

Lemma 2.1. *Let $U : H \to K$ be an unitary operator, and $\lambda \in [0,1]$. We have the following identity*

$$\Delta_\lambda(UTU^*) = U\Delta_\lambda(T)U^*, \quad \textit{for every}\ T \in \mathcal{B}(H).$$

Proof. Let $T \in \mathcal{B}(H)$. It is easy to check

$$|UTU^*| = U|T|U^* \quad \text{and} \quad |UTU^*|^\lambda = U|T|^\lambda U^*, \ \ \lambda \in [0,1].$$

Now, let $T = V|T|$ be a polar decomposition. Then

$$UTU^* = UV|T|U^* = (UVU^*)(U|T|U^*) = \tilde{V}|UTU^*|,$$

where $\tilde{V} = UVU^*$. $\tilde{V}$ is a partial isometry, $\mathcal{N}(UTU^*) = \mathcal{N}(\tilde{V})$ and hence $\tilde{V}|UTU^*|$ is the polar decomposition of UTU^*. This implies that :

$$\begin{aligned}\Delta_\lambda(UTU^*) &= |UTU^*|^\lambda \tilde{V}|UTU^*|^{1-\lambda} \\ &= U|T|^\lambda U^*\tilde{V}U|T|^{1-\lambda}U^* \\ &= U|T|^\lambda V|T|^{1-\lambda}U^* \\ &= U\Delta_\lambda(T)U^*.\end{aligned}$$

This completes the proof. ■

For $x, y \in H$, we denote by $x \otimes y$ the at most rank one operator defined by

$$(x \otimes y)u =< u, y > x \ \text{ for } \ u \in H.$$

It is easy to show that every rank one operator has the previous form and that $x \otimes y$ is an orthogonal projection, if and only if $x = y$ and $\|x\| = 1$. We have the following proposition,

Proposition 2.1. *Let $x, y \in H$ be nonzero vectors. We have*

$$\Delta_\lambda(x \otimes y) = \frac{< x, y >}{\|y\|^2}(y \otimes y) \quad \textit{for every} \ \ \lambda \in]0,1[.$$

Proof. Denote $T = x \otimes y$, then $T^*T = |T|^2 = \|x\|^2(y \otimes y) = \left(\frac{\|x\|}{\|y\|}(y \otimes y)\right)^2$ and $|T| = \sqrt{T^*T} = \frac{\|x\|}{\|y\|}(y \otimes y)$. It follows that $|T|^2 = \|x\|\|y\||T|$ and $|T|^\gamma = (\|x\|\|y\|)^{\gamma-1}|T|$ for every $\gamma > 0$.

Now, let $T = U|T|$ be the polar decomposition of T, we have

$$\begin{aligned}\Delta_\lambda(T) &= |T|^\lambda U|T|^{1-\lambda} \\ &= (\|x\|\|y\|)^{\lambda-1}(\|x\|\|y\|)^{-\lambda}|T|U|T| \\ &= \frac{1}{\|x\|\|y\|}|T|T \\ &= \frac{1}{\|y\|^2}(y \otimes y) \circ (x \otimes y) = \frac{< x, y >}{\|y\|^2}(y \otimes y).\end{aligned}$$

■

We deduce the next

Corollary 2.1. *Let R be a bounded linear operator on H and $\lambda \in]0,1[$. Suppose that*

$$\Delta_\lambda(RT) = \Delta_\lambda(TR),$$

for every rank one operator of the form $T = y \otimes y$. Then, there exists some $\alpha \in \mathbb{C}$ such that $R = \alpha I$.

Proof. Denote $A = R^*$. First, we claim that the linear operator A satisfies the property that for every $z \in H$ we either have Az is orthogonal to z (calling z being of the first kind) or Az, z are linearly dependent (calling z being of the second kind). Indeed, let $z \in H$ and $T = z \otimes z$, from the assumption and the Proposition 2.1, we have

$$< Rz, z > z \otimes z = \Delta_\lambda(Rz \otimes z) = \Delta_\lambda(z \otimes Az).$$

In the case when $< Rz, z >= 0$, then z is of the first kind. And if $< Rz, z >\neq 0$ then $Az \neq 0$, and from the last equality it follows that

$$< Rz, z > z \otimes z = \frac{< Rz, z >}{\|Az\|^2} Az \otimes Az.$$

Thus Az and z are linearly dependent.

Now, A is a scalar multiple of the identity. Indeed, on contrary assume that we have vector x which is of the first kind but not of the second kind and that we have a vector y which is of the second kind but not of the first kind. Then x, y are linearly independent. We may assume that $Ay = y$. Set $x' = Ax$. For a real number t from the unit interval and for $z_t = tx + (1-t)y$ we have $Az_t = tx' + (1-t)y$. It is clear that the equation $< Az_t, z_t >= t(1-t)\big(< x', y > + < y, x > \big) + (1-t)^2\|y\|^2 = 0$ has at most one solutions $t_1 \in]0, 1[$. Also, with the fact that x, y are linearly independent, then Az_t, z_t are linearly independent for all $t \in]0, 1[$ except for at most one $t \in]0, 1[$. So, for example, for small enough positive t the vector z_t does not of the first kind nor of the second.

This shows that either have that Az is orthogonal to z for all vectors z or we have Az, z are linearly dependent for all vectors z. In the first case we have that $A = 0$, in the second one A is a scalar multiple of the identity. In any way A is a scalar multiple of the identity. Thus $R = A^* = \alpha I$ for some $\alpha \in \mathbb{C}$. ∎

The following lemma, provides a criterion for an operator to be positive through its λ-Aluthge transform. It will play a crucial role in the proof of Theorem 1.1.

Lemma 2.2. *Let $T \in \mathcal{B}(H)$ be an invertible operator. The following conditions are equivalent :*

(i) T is positive;

(ii) for every $\lambda \in [0, 1], \Delta_\lambda(T)$ is positive;

(iii) there exists $\lambda \in [0, 1]$ such that $\Delta_\lambda(T)$ is positive.

In particular, $\Delta_\lambda(T) = cI$ for some nonzero scalar c, if and only if $T = cI$.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are trivial. It remains to show that $(iii) \Rightarrow (i)$. Let us consider the polar decomposition $T = U|T|$ of T and assume that $\Delta_\lambda(T)$ is a positive operator. Since T invertible it follows that $|T|^{1-\lambda}$ is invertible and U is unitary. We claim that $U = I$. Indeed, let us denote $A = |T|^{2\lambda-1}$, we have

$$\begin{aligned} AU &= |T|^{2\lambda-1}U \\ &= |T|^{\lambda-1}(|T|^\lambda U|T|^{1-\lambda})|T|^{\lambda-1} \\ &= |T|^{\lambda-1}\Delta_\lambda(T)|T|^{\lambda-1}. \end{aligned}$$

This follows that $AU = |T|^{\lambda-1}\Delta_\lambda(T)|T|^{\lambda-1}$ is positive. In particular it is self adjoint. Thus $AU = (AU)^* = U^*A$ and then $UAU = A$. Therefore $(AU)^2 = A^2$. It follows that $AU = A$ since AU and A are positive. Thus $U = I$ and this gives $T = U|T| = |T|$ is positive. ∎

Remark 2.1. *The assumption T is invertible is necessary in the previous lemma. Indeed, let $T = x \otimes y$, with x, y be nonzero independent vectors such that $< x, y >\geq 0$. Using proposition 2.1, we get $\Delta_\lambda(T)$ is positive while T is not.*

Lemma 2.3. *Let $T \in \mathcal{B}(H)$ be an arbitrary operator and $P \in \mathcal{B}(H)$ be an orthogonal projection. The following are equivalent :*

(i) $\Delta_\lambda(TP) = T$ *;*

(ii) $TP = PT = T$ *and* T *is quasi-normal.*

Proof. The implication $(ii) \Rightarrow (i)$ is obvious. We show the direct implication. Consider $TP = U|TP|$ the polar decomposition of TP. Suppose that $\Delta_\lambda(TP) = T$, then

$$|TP|^\lambda U|TP|^{1-\lambda} = T \quad \text{and} \quad |TP|^{1-\lambda}U^*|TP|^\lambda = T^*. \tag{4}$$

It follows that

$$\mathcal{R}(T) \subseteq \mathcal{R}(|TP|^\lambda) \subseteq \overline{\mathcal{R}(|TP|^2)}$$

and

$$\mathcal{R}(T^*) \subseteq \mathcal{R}(|TP|^{1-\lambda}) \subseteq \overline{\mathcal{R}(|TP|^2)}.$$

In the other hand, we have $|TP|^2 = PT^*TP = P|T|^2P$. Thus $\overline{\mathcal{R}(|TP|^2)} \subseteq \mathcal{R}(P)$. Hence $\mathcal{R}(T) \subset \mathcal{R}(P)$ and $\mathcal{R}(T^*) \subset \mathcal{R}(P)$. Which implies that $PT = T$ and $PT^* = T^*$. Therefore

$$PT = TP = T \quad \text{and } T \text{ is quasi-normal.}$$

■

Proposition 2.2. *Let* Φ *be a bijective map satisfying* (2)*. Then*

$$\Phi(0) = 0.$$

Moreover, there exists a bijective function $h : \mathbb{C} \to \mathbb{C}$ *such that:*

(i) $\Phi(\alpha I) = h(\alpha)I$ *for all* $\alpha \in \mathbb{C}$.

(ii) $h(\alpha\beta) = h(\alpha)h(\beta)$ *for all* $\alpha, \beta \in \mathbb{C}$.

(iii) $h(1) = 1$ *and* $h(-\alpha) = -h(\alpha)$ *for all* $\alpha \in \mathbb{C}$.

Proof. For the first assertion, since Φ is bijective, there exists $A \in \mathcal{B}(H)$ such that $\Phi(A) = 0$. Therefore $\Phi(0) = \Delta_\lambda(\Phi(A)\Phi(0)) = 0$.

Let us show now that there exists a function $h : \mathbb{C} \to \mathbb{C}$ such that $\Phi(\alpha I) = h(\alpha)I$ for all $\alpha \in \mathbb{C}$. If $\alpha = 0$ the function h is defined by $h(0) = 0$ since $\Phi(0) = 0$. Now, suppose that α is a nonzero scalar and denote by $R = \Phi(\alpha I)$ in particular $R \neq 0$. From (2) it follows that

$$\Delta_\lambda(R\Phi(A)) = \Phi(\Delta_\lambda(\alpha A)) = \Delta_\lambda(\Phi(A\alpha I)) = \Delta_\lambda(\Phi(A)R), \tag{5}$$

for every $A \in \mathcal{B}(H)$. Since Φ is onto, then $\Delta_\lambda(RT) = \Delta_\lambda(TR)$ for every rank one operator of the form $T = y \otimes y$ from $\mathcal{B}(K)$. Since $R = \Phi(\alpha I)$ different from zero and by Corollary 2.1, there exists a nonzero scalar $h(\alpha) \in \mathbb{C}$ such that $R = \Phi(\alpha I) = h(\alpha)I$. In the other hand, Φ is bijective and its inverse Φ^{-1} satisfies the same condition as Φ. It follows that the map $h : \mathbb{C} \to \mathbb{C}$ is well defined and it is bijective.

Moreover, using again Condition (2), we get

$$h(\alpha\beta)I = \Delta_\lambda(\Phi(\alpha\beta I)) = \Delta_\lambda(\Phi(\alpha I)\Phi(\beta I)) = h(\alpha)h(\beta)I,$$

for every $\alpha, \beta \in \mathbb{C}$ and therefore h is multiplicative.

Since $(h(1))^2 = h(1)$ and h is bijective with $h(0) = 0$, we obtain $h(1) = 1$. Similarly $h(-1) = -1$, thus $h(-\alpha) = h(-1)h(\alpha) = -h(\alpha)$ for all $\alpha \in \mathbb{C}$. ■

As a direct consequence we have the following corollary :

Corollary 2.2. *Let* $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ *be a bijective map satisfying* (2)*. Then*

(i) $\Phi(I) = I$.

(ii) $\Delta_\lambda \circ \Phi = \Phi \circ \Delta_\lambda$. *In particular,* Φ *preserves the set of quasi-normal operators in both directions.*

(iii) $\Phi(\alpha A) = h(\alpha)\Phi(A)$ *for all* α *and* A *quasi-normal.*

The following lemma gives some properties of bijective maps satisfying (2).

Lemma 2.4. *Let* Φ *be a bijective map satisfying* (2)*. Then*

(1) $\Phi(A^2) = (\Phi(A))^2$ *for all* A *quasi-normal.*

(2) Φ *preserves the set of orthogonal projections.*

(3) Φ *preserves the orthogonality between the projections ;*

$$P \perp Q \Leftrightarrow \Phi(P) \perp \Phi(Q).$$

(4) Φ *preserves the order relation on the set of orthogonal projections in the both directions ;*

$$Q \leq P \Leftrightarrow \Phi(Q) \leq \Phi(P).$$

(5) $\Phi(P+Q) = \Phi(P) + \Phi(Q)$ *for all orthogonal projections* P, Q *such that* $P \perp Q$*.*

(6) Φ *preserves the set of rank one orthogonal projections in the both directions.*

Proof. (1) From (2) , we have $\Delta_\lambda((\Phi(A))^2) = \Phi(\Delta_\lambda(A^2))$ for every operator A. Let A be a quasi-normal operator; since Φ preserves the set of quasi-normal operators, we get $\Phi(A), \Phi(A^2), (\Phi(A))^2$ are quasi-normal. It follows from (3)) that $\Delta_\lambda(A^2) = A^2$ and $\Delta_\lambda(\Phi(A^2)) = \Phi(A^2)$. We deduce that

$$(\Phi(A))^2 = \Delta_\lambda((\Phi(A))^2) = \Phi(\Delta_\lambda(A^2)) = \Phi(A^2).$$

(2) Follows from the first assertion since orthogonal projections are quasi-normal.

(3) Assume that P, Q are orthogonal and denote $N = \Phi(P)$ and $M = \Phi(Q)$. From (2) we have, $\Delta_\lambda(MN) = \Delta_\lambda(NM) = 0$ and using [6, Theorem 4], we obtain

$$(MN)^2 = MNMN = 0 \quad \text{and} \quad (NM)^2 = NMNM = 0.$$

It follows that,

$$\|MN\|^2 = \|(MN)^*MN\| = \|NMN\| = \|(NMN)^2\|^{\frac{1}{2}} = \|NMNMN\|^{\frac{1}{2}} = 0$$

and similarly, $NM = 0$.

Finally Φ preserves the orthogonality between the projections.

(4) Now, assume that $Q \leq P$, then $PQ = QP = Q$. By (2) we have

$$\Delta_\lambda(\Phi(Q)\Phi(P)) = \Phi(Q).$$

By Lemma 2.3, we get $\Phi(Q)\Phi(P) = \Phi(P)\Phi(Q) = \Phi(Q)$ since $\Phi(P)$ is an orthogonal projection. Therefore $\Phi(Q) \leq \Phi(P)$. Since Φ is bijective and its inverse satisfies the same conditions as Φ, hence Φ preserves the order relation between the projections in both directions.

(5) Suppose that P, Q are orthogonal. We have $P \leq P+Q$ and $Q \leq P+Q$. Which gives $\Phi(P) \leq \Phi(P+Q)$ and $\Phi(Q) \leq \Phi(P+Q)$. From $\Phi(P) \perp \Phi(Q)$, it follows that

$$\Phi(P) + \Phi(Q) \leq \Phi(P+Q).$$

Since Φ^{-1} satisfies the same assumptions as Φ, we have

$$\begin{array}{rcl} \Phi(P+Q) & = & \Phi[\Phi^{-1}(\Phi(P)) + \Phi^{-1}(\Phi(Q))] \\ & \leq & \Phi[\Phi^{-1}(\Phi(P) + \Phi(Q))] \\ & = & \Phi(P) + \Phi(Q). \end{array}$$

Finally $\Phi(P+Q)=\Phi(P)+\Phi(Q)$.

(6) Let $P=x\otimes x$ be a rank one projection. We claim that $\Phi(P)$ is a non zero minimal projection. Indeed, let $y\in K$ be an unit vector such that $y\otimes y\leq\Phi(P)$. Thus $\Phi^{-1}(y\otimes y)\leq P$. Since P is a minimal projection and $\Phi^{-1}(y\otimes y)$ is a non zero projection, then $\Phi^{-1}(y\otimes y)=P$. Therefore $\Phi(P)=y\otimes y$ is a rank one projection. ■

We now prove the following lemma which is needed in the proof of our result.

Lemma 2.5. *Let Φ be a bijective map satisfying (2). Let $P=x\otimes x, Q=x'\otimes x'$ be rank one projections such that $P\perp Q$. Then*

$$\Phi(\alpha P+\beta Q)=h(\alpha)\Phi(P)+h(\beta)\Phi(Q)$$

for every $\alpha,\beta\in\mathbb{C}$.

Proof. If $\alpha=0$ or $\beta=0$ the result is trivial. Suppose that $\alpha\neq 0$ and $\beta\neq 0$. Clearly $\alpha P+\beta Q$ is normal, hence $\Phi(\alpha P+\beta Q)$ is quasi-normal. By Condition (2) we get

$$\begin{aligned}\Phi(\alpha P+\beta Q) &= \Delta_\lambda(\Phi(\alpha P+\beta Q))\\ &= \Phi(\Delta_\lambda(\alpha P+\beta Q))\\ &= \Phi\Big(\Delta_\lambda\big((\alpha P+\beta Q)(P+Q)\big)\Big)\\ &= \Delta_\lambda\big(\Phi(\alpha P+\beta Q)\Phi(P+Q)\big)\\ &= \Phi(\alpha P+\beta Q)\Phi(P+Q).\end{aligned}$$

Since $\Phi(P+Q)=\Phi(P)+\Phi(Q)$ is a an orthogonal projection, hence by Lemma 2.3

$$\begin{aligned}\Phi(\alpha P+\beta Q) &= \Phi(\alpha P+\beta Q)(\Phi(P)+\Phi(Q))=(\Phi(P)+\Phi(Q))\Phi(\alpha P+\beta Q)\\ &= (\Phi(P)+\Phi(Q))\Phi(\alpha P+\beta Q)(\Phi(P)+\Phi(Q)).\end{aligned}$$

Denote by $T=\Phi(\alpha P+\beta Q)$. We write $\Phi(x\otimes x)=y\otimes y$ and $\Phi(x'\otimes x')=y'\otimes y'$ with $y\perp y'$, since Φ preserves orthogonality and rank one projections. We have,

$$T=(y\otimes y+y'\otimes y')T(y\otimes y+y'\otimes y').$$

Hence

$$T=<Ty,y>y\otimes y+<Ty',y>y\otimes y'+<Ty,y'>y'\otimes y+<Ty',y'>y'\otimes y'. \tag{6}$$

We show that $<Ty',y>=<Ty,y'>=0$ by using (2)

$$\begin{aligned}\Delta_\lambda(\Phi(\alpha P+\beta Q)\Phi(P)) &= \Phi(\Delta_\lambda((\alpha P+\beta Q)P))\\ &= \Phi(\alpha P)=h(\alpha)\Phi(P).\end{aligned}$$

In other terms, we write

$$\Delta_\lambda(Ty\otimes y)=\Delta_\lambda(y\otimes T^*y)=h(\alpha)y\otimes y.$$

Since $h(\alpha)\neq 0$, then $T^*y\neq 0$. By Proposition 2.1 follows that

$$<Ty,y>y\otimes y=\frac{<y,T^*y>}{\|T^*y\|^2}T^*y\otimes T^*y=h(\alpha)y\otimes y.$$

Therefore $<Ty,y>=h(\alpha)$ and $T^*y=\overline{h(\alpha)}y$. Using (6) we deduce

$$T^*y=<T^*y,y>y+<T^*y,y'>y'=\overline{h(\alpha)}y.$$

It follows that $<Ty',y>=<T^*y,y'>=0$.

By similar arguments we get $<Ty',y'>=h(\beta)$ and $<Ty,y'>=0$. Again (6) implies that

$$\Phi(\alpha P+\beta Q)=T=h(\alpha)y\otimes y+h(\beta)y'\otimes y'=h(\alpha)\Phi(P)+h(\beta)\Phi(Q).$$

■

Now, we are in a position to prove our main result

Proof of Theorem 1.1. The "only if" part is an immediate consequence of Lemma 2.1.

We show the "if" part. Assume that $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is bijective and satisfies (2). The proof of theorem is organized in several steps.

Step 1. For every $A \in \mathcal{B}(H)$, we have

$$< \Phi(A)y, y >= h(< Ax, x >) \text{ for all unit vectors } x, y \text{ such } \Phi(x \otimes x) = y \otimes y. \tag{7}$$

Let $x, y \in H$ be unit vectors such that $\Phi(x \otimes x) = y \otimes y$. From (2), we obtain

$$\begin{aligned} \Delta_\lambda(\Phi(A)y \otimes y) &= \Delta_\lambda(\Phi(A)\Phi(x \otimes x)) \\ &= \Phi(\Delta_\lambda(A(x \otimes x))) \\ &= \Phi(\Delta_\lambda(Ax \otimes x)). \end{aligned}$$

Using Proposition 2.1, we get

$$< \Phi(A)y, y > y \otimes y = \Phi(< Ax, x > x \otimes x) = h(< Ax, x >)y \otimes y.$$

It follows that

$$< \Phi(A)y, y >= h(< Ax, x >).$$

Step 2. The function h is additive.

Let $P = x \otimes x, Q = x' \otimes x'$ are rank one projections such that $P \perp Q$ and $\alpha, \beta \in \mathbb{C}$. Denote by $z = \frac{1}{\sqrt{2}}(x+x')$, then $\|z\| = 1$ and $\|Pz\|^2 = \|Qz\|^2 = \frac{1}{2}$. Note $z \otimes z$ is rank one projection, then there exist an unit vector $u \in K$ such that $\Phi(z \otimes z) = u \otimes u$. We take $A = \alpha P + \beta Q$ in the identity (7), we get that

$$\begin{aligned} < \Phi(\alpha P + \beta Q)u, u > &= h(< \alpha Pz + \beta Qz, z >) \\ &= h(\alpha\|Pz\|^2 + \beta\|Qz\|^2) \\ &= h(\frac{1}{2})h(\alpha + \beta). \end{aligned}$$

Thus

$$< \Phi(\alpha P + \beta Q)u, u >= h(\frac{1}{2})h(\alpha + \beta). \tag{8}$$

In the other hand, by Lemma 2.5 we have

$$\Phi(\alpha P + \beta Q) = \Phi(\alpha P) + \Phi(\beta Q) = h(\alpha)\Phi(P) + h(\beta)\Phi(Q).$$

And therefore

$$\begin{aligned} < \Phi(\alpha P + \beta Q)u, u > &= < \Phi(\alpha P)u + \Phi(\beta Q)u, u > \\ &= < \Phi(\alpha P)u, u > + < \Phi(\beta Q)u, u > \\ &= h(< \alpha Pz, z >) + h(< \beta Qz, z >) \\ &= h(\alpha\|Pz\|^2) + h(\beta\|Qz\|^2) \\ &= h(\frac{1}{2})(h(\alpha) + h(\beta)). \end{aligned}$$

Using (8) and the preceding equality, it follows that

$$h(\frac{1}{2})h(\alpha + \beta) = h(\frac{1}{2})(h(\alpha) + h(\beta)).$$

Now $h(\frac{1}{2}) \neq 0$ gives

$$h(\alpha + \beta) = h(\alpha) + h(\beta).$$

Step 3. h is continuous.

Let $\mathcal{E}$ be a bounded subset in $\mathbb{C}$ and $A \in \mathcal{B}(H)$ such that $\mathcal{E} \subset W(A)$.

By (7),

$$h(\mathcal{E}) \subset h(W(A)) = W(\Phi(A))$$

Now , $W(\Phi(A))$ is bounded and thus h is bounded on the bounded subset. With the fact that h is additive and multiplicative, it then follows that h is continuous (see, for example, [10]). We derive that h is a continuous automorphism over the complex field $\mathbb{C}$. It follows that h is the identity or the complex conjugation map.

Step 4. The map Φ is linear or anti-linear.

Let $y \in K$ and $x \in H$ be two unit vectors, such that $y \otimes y = \Phi(x \otimes x)$. Let $\alpha \in \mathbb{C}$ and $A, B \in \mathcal{B}(H)$ be arbitrary. Using (7), we get

$$\begin{aligned} < \Phi(A+B)y, y > &= h(< (A+B)x, x >) \\ &= h(< Ax, x > + < Bx, x >) \\ &= h(< Ax, x >) + h(< Bx, x >) \\ &= < \Phi(A)y, y > + < \Phi(B)y, y > \\ &= < (\Phi(A) + \Phi(B))y, y >, \end{aligned}$$

and

$$< \Phi(\alpha A)y, y > \quad = \quad h(< \alpha Ax, x >) = h(\alpha)h(< Ax, x >) = h(\alpha) < \Phi(A)y, y > .$$

Therefore

$$< \Phi(A+B)y, y >=< (\Phi(A) + \Phi(B))y, y > \quad \text{and} < \Phi(\alpha A)y, y >= h(\alpha) < \Phi(A)y, y >,$$

for all unit vectors $y \in K$. It follows that $\Phi(A+B) = \Phi(A) + \Phi(B)$ and $\Phi(\alpha A) = h(\alpha)\Phi(A)$ for all $A, B \in \mathcal{B}(H)$. Therefore Φ is linear or anti-linear since h is the identity or the complex conjugation.

Step 5. There exists an unitary operator $U \in \mathcal{B}(H, K)$, such that $\Phi(A) = UAU^*$ for every $A \in \mathcal{B}(H)$.

Let $A \in \mathcal{B}(H)$ be invertible. By (2), we have

$$\Delta_\lambda(\Phi(A)\Phi(A^{-1})) = \Delta_\lambda(\Phi(A^{-1})\Phi(A)) = \Phi(\Delta_\lambda(I)) = I.$$

By Lemma 2.2, we get that

$$\Phi(A)\Phi(A^{-1}) = \Phi(A^{-1})\Phi(A) = I.$$

It follows that $\Phi(A)$ is also invertible and $(\Phi(A))^{-1} = \Phi(A^{-1})$. Therefore Φ preserves the set of invertible operators. By [5, Corollary 4.3], there exists a bounded linear and bijective operator $V : H \to K$ such that Φ takes one of the following form

$$\Phi(A) = VAV^{-1} \quad \text{for all} \;\; A \in \mathcal{B}(H) \tag{9}$$

or

$$\Phi(A) = VA^*V^{-1} \quad \text{for all} \;\; A \in \mathcal{B}(H) \tag{10}$$

In order to complete the proof we have to show that V is unitary and Φ has form (9).

First, we show that $V : H \to K$ in (9) (or in (10)) is necessarily unitary. Indeed, let $x \in H$ be a unit vector. We know that $x \otimes x$ is an orthogonal projection, hence $\Phi(x \otimes x) = Vx \otimes (V^{-1})^*x$ is also an orthogonal projection. It follows that $(V^{-1})^*x = Vx$ for all unit vector $x \in H$ and then $(V^{-1})^* = V$. Therefore V is unitary.

Seeking contradiction, we suppose that (10) holds. Multiplying (10) by V^* left and by V right, since Φ commutes with Δ_λ, we obtain

$$\Delta_\lambda(A^*) = (\Delta_\lambda(A))^*, \quad \text{for every } A \in \mathcal{B}(H). \tag{11}$$

Let us consider $A = x \otimes x'$ with x, x' are unit independent vectors in H. $A^* = x' \otimes x$ and by Proposition 2.1, we have

$$\Delta_\lambda(A) =< x, x' > (x' \otimes x') \text{ and } \Delta_\lambda(A^*) =< x', x > (x \otimes x),$$

which contradicts (11). This completes the proof.

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References

[1] A. Aluthge. On p-hyponormal operators for $0 < p < 1$. *Integral Equations Operator Theory*, 13(3):307–315, 1990.

[2] T. Ando and T. Yamazaki. The iterated Aluthge transforms of a 2-by-2 matrix converge. *Linear Algebra Appl.*, 375:299–309, 2003.

[3] J. Antezana, P. Massey, and D. Stojanoff. λ-Aluthge transforms and Schatten ideals. *Linear Algebra Appl.*, 405:177–199, 2005.

[4] F. Botelho, L. Molnár, and G. Nagy. Linear bijections on von Neumann factors commuting with λ-Aluthge transform. *Bull. Lond. Math. Soc.*, 48(1):74–84, 2016.

[5] N. Boudi and M. Mbekhta. Additive maps preserving strongly generalized inverses. *J. Operator Theory*, 64(1):117–130, 2010.

[6] S. R. Garcia. Aluthge transforms of complex symmetric operators. *Integral Equations Operator Theory*, 60(3):357–367, 2008.

[7] I. B. Jung, E. Ko, and C. Pearcy. Aluthge transforms of operators. *Integral Equations Operator Theory*, 37(4):437–448, 2000.

[8] I. B. Jung, E. Ko, and C. Pearcy. Spectral pictures of Aluthge transforms of operators. *Integral Equations Operator Theory*, 40(1):52–60, 2001.

[9] I. B. Jung, E. Ko, and C. Pearcy. The iterated Aluthge transform of an operator. *Integral Equations Operator Theory*, 45(4):375–387, 2003.

[10] R. R. Kallman and F. W. Simmons. A theorem on planar continua and an application to automorphisms of the field of complex numbers. *Topology Appl.*, 20(3):251–255, 1985.

[11] K. Okubo. On weakly unitarily invariant norm and the Aluthge transformation. *Linear Algebra Appl.*, 371:369–375, 2003.

[12] T. Yamazaki. An expression of spectral radius via Aluthge transformation. *Proc. Amer. Math. Soc.*, 130(4):1131–1137 (electronic), 2002.